Journal of Approximation Theory **94**, 300–305 (1998)
Article No. AT983193

Counterexample to a Problem on Tensor Product Approximation

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Communicated by Manfred v. Golitschek

Received November 4, 1996; accepted in revised form June 4, 1997

Answering a conjecture of M. von Golitschek in the negative, a compact set K is constructed on the plane such that not every continuous function on K can be uniformly approximated by continuous functions of the form $g(x)+h(y)$, and yet K does not contain a closed path of points with consecutive points connected with alternatively horizontal and vertical segments. © 1998 Academic Press

Let K be a compact set on the plane. It is a fascinating problem—connected with the geometry of K—to determine what continuous functions $F(x, y)$ on K can be approximated by tensor-sum functions of the form $g(x)+h(y)$ with continuous g and h (for tensor product spaces in general see [1], for many different applications of this type of approximation see the paper [2] by M. van Golitschek). In particular, when is it true that every continuous F on K can be uniformly approximated by such tensor sums? It is easy to see that if K contains a sequence of distinct points $P_1(x_1, y_1)$, $P_2(x_1, y_2)$, $P_3(x_2, y_2)$, $P_4(x_2, y_3)$, $P_5(x_3, y_3)$, ..., $P_{2k-1}(x_k, y_k)$, $P_{2k}(x_k, y_1)$, i.e., for which the line segments $\overline{P_j P_{j+1}}$ $(P_{2k+1}=P_1)$ are alternatively vertical and horizontal, then there are functions F that are not approximable. In fact, it is enough to note that for any function $\mathscr{F}(x, y)=g(x)+h(y)$ the sum

$$\sum_{j=1}^{k} (\mathscr{F}(P_{2j})-\mathscr{F}(P_{2j-1}))$$

is zero, so, e.g., if $F(P_1)=1$ and $F(P_j)=0$, $j=2, ..., 2k$, then F cannot be approximated with error less than $1/2k$ by any function $\mathscr{F}(x, y)$.

* Research was supported in part by the National Science Foundation, DMS-9415657 and by the Hungarian Academy of Sciences, Grant 96-328.

0021-9045/98 $25.00

Let us call a sequence $P_1, ..., P_{2k}$ with the above property a closed twisting path (closed means that from the last point we get back to the first one; and by "twisting" we are just referring to the vertical-horizontal property of the sequence). In [2] M. von Golitschek conjectured that the existence of a closed twisting path is the only obstacle that prevents approximation, i.e., he made the

Conjecture. If K does not contain a closed twisting path, then every continuous function on K can be uniformly approximated by functions of the form $g(x)+h(y)$ with continuous g and h.

In this note we show that this is not true. We should like to point out that if the continuity of the functions $g(x)$ and $h(y)$ are not required, then approximation (actually representation) is possible (provided there is no closed twisting path). In fact, let for $P, Q \in K$ be $P \sim Q$ if P can be reached from Q by a twisting path (not necessarily of even length). Then $\sim$ is an equivalence relation, and points in different equivalence classes have different x and y coordinates. Therefore, it is enough to show that any function F is of the form $g(x)+h(y)$ on each equivalence class H. Let $P \in H$. Since there is no closed twisting path in K, every point in H can be reached from P via a unique twisting path. Along each such path we use the required identity $F(x, y) = g(x) + h(y)$ to define g and h, and the absence of closed twisting paths guarantees that this process will never yield contradictory values. Note that this procedure will generally result in unbounded g and h. This is not accidental, for a result of M. von Golitschek and W. A. Light [3] says that for continuous functions $f(x, y)$ approximation by tensor sums of the form $g(x)+h(y)$ with continuous g and h is equivalent to approximation by tensor sums of the form $g(x)+h(y)$ with *bounded* g and h.

We are going to construct a compact set K such that it does not contain a closed twisting path, yet it contains an infinite twisting path $P_1, P_2, ...,$ $P_n, ...,$ such that the distance between consecutive points P_j and P_{j+1} is at least 1 (note that without this second requirement a pair segment with one common endpoint and of inclination angle smaller than 90 degree would satisfy the requirements for K, but such a K is not appropriate for resolving the conjecture). After the construction we shall show that this is enough; the conjecture is not valid for this K.

Let C be the usual triadic Cantor set. The points of C can be uniquely written in base 3 in the form

$$\alpha = 0, \alpha_1 \alpha_2, ...,$$

where the digits α_i are 0 or 2.

First we construct a continuous $f: C \to C$ bijection such that no iterant of f has a fixed point. Let, e.g., $f(1) = 0$, and if in the triadic expansion of

α the first zero number is α_n, then let $f(\alpha)_j = 0$ for every $1 \leqslant j < n$, $f(\alpha)_n = 2$, and $f(\alpha)_j = \alpha_j$ otherwise. In other words, we annihilate the 2's before the first zero, write 2 instead of the first zero, and leave all other digits unchanged. For example, $f(0) = 2/3$, $f(1/3) = 1$, and in general $f(x) = x + 2/3$ if $x \in [0, 1/3] \cap C$, $f(x) = x - 4/9$ if $x \in [2/3, 7/9] \cap C$, etc. It is clear that f is continuous. It is also easy to see that f is a bijection of C onto itself. We show that no iterant of f has a fixed point.

Let the zero digits in the triadic expansion of $\alpha \in C$ be at the $(n_1 + 1)$st $(n_2 + 1)$st, etc., places (in this order), and first let us assume that there are infinitely many of them. Then

$$\alpha = 0, \overbrace{2 \cdots 2}^{n_1} 0\, \alpha_{n_1+2} \cdots,$$

$$f(\alpha) = 0, \overbrace{0 \cdots 0}^{n_1} 2\, \alpha_{n_1+2} \cdots,$$

$$f(f(\alpha)) := f^{(2)}(\alpha) = 0, 2\, \overbrace{0 \cdots 0}^{n_1 - 1} 2\, \alpha_{n_1+2} \cdots,$$

$$f^{(3)}(\alpha) = 0, 02\, \overbrace{0 \cdots 0}^{n_1 - 2} 2\, \alpha_{n_1+2} \cdots,$$

$$f^{(4)}(\alpha) = 0, 22\, \overbrace{0 \cdots 0}^{n_1 - 2} 2\, \alpha_{n_1+2} \cdots,$$

$$\vdots$$

$$f^{(2^{n_1})}(\alpha) = 0, \overbrace{2 \cdots 2}^{n_2} 0\, \alpha_{n_2+2} \cdots,$$

therefore, the $(n_1 + 1)$st digit in $f^{(j)}(\alpha)$ equals 2 for every $1 \leqslant j \leqslant 2^{n_1}$, while the same digit is 0 in the expansion of α, so for such j we have $f^{(j)}(\alpha) \neq \alpha$. Now

$$f^{(2^{n_1}+1)}(\alpha) = 0, \overbrace{0 \cdots 0}^{n_2} 2\, \alpha_{n_2+2} \cdots,$$

$$f^{(2^{n_1}+2)}(\alpha) = 0, 2\, \overbrace{0 \cdots 0}^{n_2 - 2} 2\, \alpha_{n_2+2} \cdots,$$

$$\vdots$$

$$f^{(2^{n_1}+2^{n_2})}(\alpha) = 0, \overbrace{2 \cdots 2}^{n_3} 0\, \alpha_{n_3+2} \cdots,$$

therefore the (n_2+1)st digit in $f^{(j)}(\alpha)$ equals 2 for every $2^{n_1} < j \leqslant 2^{n_1} + 2^{n_2}$, while the same digit is 0 in α, so for such j's, $f^{(j)}(\alpha) \neq \alpha$. Continuing this procedure we obtain that $f^{(j)}(\alpha) \neq \alpha$ for any j.

If there are only finitely many zeros in the expansion of α, the argument is very similar.

Let now K consist of the graph of f and of the segments

$$\overline{(2,0)(3,1)}, \qquad \overline{(2,-2)(3,-1)}, \qquad \overline{(0,-2)(1,-1)}.$$

Let us denote these parts (in the given order) of K by K_1, K_2, K_3, and K_4 (see Fig. 1). We shall see that this K is a set we are looking for.

First we show that K does not contain a closed twisting path. In fact, it follows by the fact that f is a bijection of C onto itself that any twisting part is uniquely determined once the starting point and the direction (horizontal or vertical) of the starting segment is given (if we start from K_2, K_3, or K_4, then it may happen that the path halts after one or two steps). It is easy to see that if $P_1, \ldots, P_m$ is a twisting path for which $P_j(x_j, y_j) \in K_1$ and the $\overline{P_j P_{j+1}}$ segment is horizontal, then the x-coordinate of the point P_{j+4} coincides with the y-coordinate of the point P_j, that is, $x_{j+4} = y_j$. But

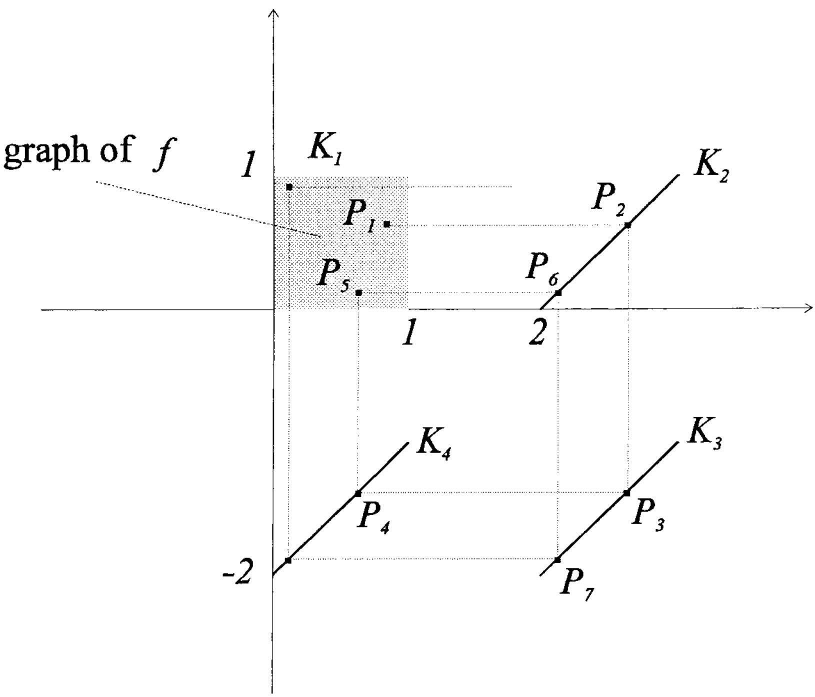

FIGURE 1

$y_j = f(x_j)$, therefore we arrive at the relation $x_{j+4} = f(x_j)$. Now the return of the path into itself would yield a fixed point of an iterant of f and there is no such fixed point. Similarly, if the segment $\overline{P_j P_{j+1}}$, $P_j \in K_1$ is vertical, then writing the sequence in reversed order we can apply the preceding argument.

Finally, we show that there is an infinite twisting path with consecutive points at least 1 apart. In fact, starting from any point of K_1 we can go alternatively horizontally and vertically first to a point of K_2, then to a point of K_3, K_4, and finally again to a point in K_1 (apply the above observation on the coordinates of P_j and P_{j+4}, and the fact that f maps the Cantor set into itself); and from here the procedure can be iterated resulting in an infinite twisting path.

Finally, let $F(x, y) = 1$ on K_1, and $F(x, y) = 0$ on $K_2 \cup K_3 \cup K_4$. We claim that for $\varepsilon = 1/5$ there is no function $\mathscr{F}(x, y) = g(x) + h(y)$ that is closer than ε to $F(x, y)$ in the supremum norm on K. In fact, suppose

$$|F(x, y) - \mathscr{F}(x, y)| \leqslant \tfrac{1}{5}$$

for all $(x, y) \in K$. Let $P_1(x_1, y_1), P_2(x_2, y_2), \ldots$ be the infinite twisting path starting from K_1 considered above. Then $P_{4k+1} \in K_1$ for all k, and

$$\mathscr{F}(P_{4(k+1)+1}) - \mathscr{F}(P_{4(k+1)}) + \mathscr{F}(P_{4(k+1)-1}) - \mathscr{F}(P_{4(k+1)-2})$$

differs from

$$F(P_{4(k+1)+1}) - F(P_{4(k+1)}) + F(P_{4(k+1)-1}) - F(P_{4(k+1)-2}) = 1$$

by at most 4/5, so

$$\mathscr{F}(P_{4(k+1)+1}) - \mathscr{F}(P_{4(k+1)}) + \mathscr{F}(P_{4(k+1)-1}) - \mathscr{F}(P_{4(k+1)-2}) \geqslant \tfrac{1}{5}.$$

However, this last expression is nothing else than

$$h(y_{4(k+1)+1}) - h(y_{4(k+1)}) + h(y_{4(k+1)-1}) - h(y_{4(k+1)-2}),$$

and we also know that

$$y_{4(k+1)} = y_{4(k+1)-1} \qquad \text{and} \qquad y_{4(k+1)-2} = y_{4k+1},$$

therefore we finally arrive at

$$h(y_{4(k+1)+1}) - h(y_{4k+1}) \geqslant \tfrac{1}{5}.$$

By adding these together for $k = 0, 1, ..., N-1$ we obtain

$$h(y_{4N+1}) - h(y_1) \geqslant \frac{N}{5},$$

which for $N \to \infty$ is in a clear contradiction to the boundedness of the function h.

This contradiction proves that F cannot be approximated.

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